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Totally Nonholonomic Distributions and Sub-Riemannian Structures

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Abstract

In this paper, we study totally non-holonomic distributions and sub-Riemannian structure on $\mathbb{R}^n$ $(n = 1, 2, 3, 4, \dots)$ and on the manifolds of not null curvature such as S^n and
T^n with $n \geq 3$, in order to look at this structure on a manifold like a restricting of a Riemannian metric on a totally non-holonomic distribution of this manifold.
On $\mathbb{R}^n$, it always possible to construct this structure thanks to her multitude of smooth distributions and vector fields.
On the manifolds of not null curvature, we de ne this structure on the kernel of a nondegenerate 1-form or of contact form on this manifold if the manifold has the contact structure.

Keywords : ***Totally nonholonomic distributions; Horizontal paths; Degree of nonholonomy; manifold of not null curvature; Sub-Riemannian Structure.***

1 INTRODUCTION

The study of sub-Riemannian geometry is got foundation on Riemannian geometry. It finds its interest in various fields of mathematics such as the theory of second order differential operators, stochastic differential equations and the diffusion in mechanics of nonholonom constraints. [4, 13]

This work studies the sub-Riemannian structure on different types of spaces namely, at space $(\mathbb{R}^n)$ and curved spaces (with not null curvature) M, looking at the possibility of define it on these types of spaces.
On flat spaces $\mathbb{R}^n$, let consider $\mathcal{F}$ a family of smooth vector fields, $\mathcal{F}$ defines a completely nonholonomic distribution on a space of dimension $n \geq 2$, if it has at least two fields X, Y of which the Lie bracket $[X, Y] \notin \mathcal{F}$ and completes the family, and so on until reaching a local frame of the tangent

bundle of space. Thus for any restriction of a Riemannian metric on such a family $\mathcal{F}$ we define a sub-Riemannian structure on $\mathbb{R}^n$.

On the other hand on a curved space M, to obtain the sub-Riemannian structure, we consider a non-degenerate 1-form α on the space of vector fields $\chi(M)$ (i.e. a map $\alpha : \chi(M) \longrightarrow C\infty(M)$) and let us admit that the kernel of α is a totally non-holonomic distribution (theorem 2) [1] because the kernel $Ker\alpha$ is a distribution on M (a subspace of the tangent space to M) of co-dimension 1. And with this non-degenerate 1-form on M, we obtain the vector fields X_i of $Ker\alpha$, therefore the duality product $< \alpha, X_i >$ defines a metric on the distribution $Ker\alpha$.

In particular, we show that if the space M is a contact manifold, then we always have a sub-Riemannian structure on M especially since the contact form α on M which is a non-degenerate 1-form, is a hyperplane to each point of M, therefore a distribution on M.

2. RESEARCH METHODS

2.1 Bundles and Sections

1. Let M a manifold of dimension n, a vector bundle over M is the triplet (E, π, M) where :

   (i) $\pi : E \longrightarrow M$ a surjective projection such that $\forall x \in M, \pi^{-1}(x) = E_x \equiv$ is a vector space of dimension n called Fibre of E

   (ii) E a bundle on M, is a manifold of dimension $n + m,\ m \in N$, called **Total space of base M**.

   (iii) If $m = n$, then $dimE = n + n = 2n$ (case of tangent bundle)

2. The Section of a bundle E is the function $S : M \longrightarrow E$ such as $\pi oS = id_M$.

Example 1

If $M = S^1$ the circle of 1-dim, then the vector bundle of base S^1 can be represented by the following figure :

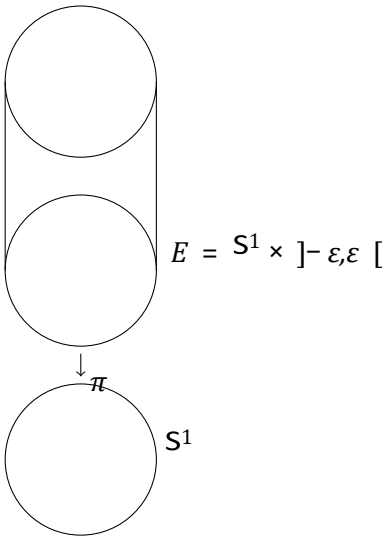


Fig 1.

2.1.1 Tangent, cotangent and Mixed bundles

1. Let M be a manifold, a bundle E of M is said to be tangent bundle to M denoted TM if its section is a vector field X, i.e. :

$$X : M \longrightarrow TM, \qquad p \longrightarrow (p, Xp)$$

with $X_p \in T_pM \equiv$ tangent vector space to M at point p such as :

$$\begin{aligned} TM &= \cup_{p\in M} T_pM \\ &= \cup_{p\in M} \{p\} \times T_pM \\ &= \cup_{p\in M} \{(p, X_p) \mid X_p \in T_pM\} \text{ . [12]} \end{aligned} \tag{1}$$

The tangent bundle TM of a manifold M is a collection of all tangent spaces of M.

Definition 1
A vector field on a manifold M is a derivation in $C^\infty(M)$ i.e., the map :

$$X : C^\infty(M) \longrightarrow C^\infty(M), \qquad f \longrightarrow X(f).$$

where $X(f)$ is the derivative of the function f.

Note : We noted by $\chi(M)$ or $\Gamma(TM)$, the set of all vector fields X on M.

Example 2

If M = R, then the tangent bundle to R can be represented by the following figure:

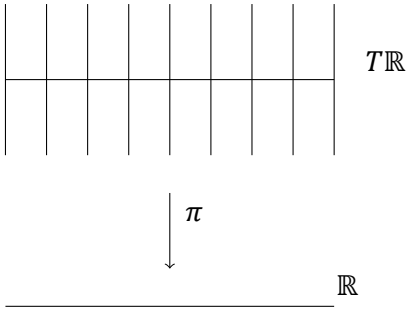


Fig 2.

2. A bundle E of M is said to be cotangent bundle of M denoted T^*M, if its section is the differential 1-form ω, i.e. :

$$\omega : M \longrightarrow T^*M, \qquad p \longrightarrow (p, \omega_p)$$

with $\omega_p \in T_p^*M \equiv$ dual space of T_pM for $p \in M$ such as :

$$T^*M = \cup_{p\in M} T_p^*M$$

(2)

.

Definition 2

A differential 1-form α on a manifold M is a dual map of $\chi(M)$ in $C^\infty(M)$ i.e.:

$$\alpha : \chi(M) \longrightarrow C^\infty(M), \qquad X \longrightarrow \alpha(X)$$

with $\alpha(X)$ a quadratic form on tangent vector space of M.

3. A bundle E of M is said to be mixed bundle of type (p, q), denoted T_q^pM if its section is the tensor field Ω, i.e. :

$$\Omega : M \longrightarrow T_q^pM$$

locally given by with

$$\Omega = \Omega^{i_1 \dots i_p}_{j_1 \dots j_q}$$

$$T_q^pM = \cup_{x\in M} \left(T_q^pM\right)_x. \qquad (3$$

3. RESULTS AND DISCUSSION

3. 1 Distributions on a manifold

1. Let M be a smooth manifold of dimension n. A distribution Δ on M is a tangent subbundle of (TM) of rank less or equal to rank of (TM) such as to each point x of M, $\Delta(x)$ is a tangent subspace of M of dimension less than or equal to $n = dim T_x M$.
   Therefore :
   i. $\Delta_x \sqsubseteq T_x M$, for $x \in M$.
   ii. $\Delta = \cup_{x \in M} \Delta_x \sqsubseteq \cup_{x \in M} T_x M = TM$.
2. A distribution Δ of rank $m \leq n$ on a smooth manifold M of dimension n is locally given by : for $x \in M$,

$$\Delta(x) = spam\{X^1(x), X^2(x), \cdots, X^m(x)\}$$

.

Remark 1
If $m = n$, then $\Delta_x = T_x M$.

Examples 3

1) If $M = \mathbb{R}$, then unique distribution on $\mathbb{R}$ is :

$$\Delta(x) = spam\left\{X_x = f(x)\frac{\partial}{\partial x}\right\} = T\mathbb{R}, \ \forall f \in C^\infty(\mathbb{R})$$

2) If $M = \mathbb{R}^2$, then

$$\Delta(x,y) = spam\left\{X^1_{(x,y)} = \frac{\partial}{\partial x}, X^2_{(x,y)} = \frac{\partial}{\partial y}\right\}, \ \forall (x,y) \in \mathbb{R}^2$$

is a distribution on $\mathbb{R}^2$.
These vector fields X^1, X^2 can be written by the coordinates like :

$$X^1 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad X^2 = \begin{pmatrix}0\\1\end{pmatrix}$$

Spaces $\Delta(x,y) = eng\left\{X_{(x,y)} = f(x,y)\frac{\partial}{\partial x}\right\}$ or $\Delta(x,y) = eng\left\{X_{(x,y)} = g(x,y)\frac{\partial}{\partial y}\right\}$
$\forall f,g \in C^{\infty}(\mathbb{R}^2)$. And others, are also the distributions on $\mathbb{R}^2$.

3) Let M,N two C^k-manifold and ϕ tangent map

$$\phi : T_xM \longrightarrow T_{f(x)}N$$

then $Ker\phi$ is a distribution on M of co-dimension 1.

4) All hyperplane on M is a distribution on M of co-dimension 1, ex. $Ker\phi$.

3.1.1 Types of distributions

1. **Smooth Distribution** : A distribution Δ of rank m is said to be smooth if to each point $x \in M$, there is a neighborhood v_x of x in M and m smooth vector fields $X_x^1,\ X_x^2 \ldots, X_x^m$ linearly independent on v_x, (i.e. to the coordinates of v_x). [1, 3]
   In other hand, $\forall y \in v_x$,
   $$\Delta(y) = spam\{X_x^1(y), X_x^2(y), \cdots, X_x^m(y)\}$$
   with $\left(X_x^i(y)\right)_i$ a free family.
2. We call "**local frame**" of a distribution Δ, a any family of smooth vector fields completive in v_x.
3. T_xM is the **trivial distribution** of M. Such a distribution does not admit a null section.
4. **Integrable distribution** : A distribution Δ is said to be integrable if
   $$\forall X^i, X^j \in \Gamma(\Delta),\ [X^i,X^j] \in \Gamma(\Delta)\ \ ou\ [X^i,X^j] = 0,\ \forall i \neq j. \qquad (4)$$
   Note that $[X,X] = 0$.

Example 4
The distributions of examples 3 (1 and 2) given above are smooth on any part of $\mathbb{R}$ and
$\mathbb{R}^2$ respectively.

Remark 2
Any family of smooth vector fields can be completed using the "Lie bracket" of its fields to obtain a local frame on M, similarly a non-smooth distribution can be completed using the Lie bracket to obtain either a smooth distribution or global bundle on M.

Example 5
Let's assume on $\mathbb{R}^4$ the family $\tau = \{X^1, X^2\}$ is not free, then it can be completed by $\Delta = \{X^1, X^2, [X^1, X^2]\}$ and become free, or smooth.

Example 6
The distributions given in example 3 (1 and 2) and T_xM are integrable.
Because in these distributions $\Delta, \forall X^i, X^j \in \Gamma(\Delta),\ [X^i, X^j] \in \Gamma(\Delta)\ ou\ [X^i, X^j] = 0$.

Example 7
In $\mathbb{R}^3$, for any point $(x, y, z) \in \mathbb{R}^3$, the distribution defined by

$$\Delta(x, y, z) = spam\{X(x, y, z); Y(x, y, z)\}, \qquad \forall (x, y, z) \in \mathbb{R}^3$$

with

$$X = \partial_x \qquad \text{and} \qquad Y = \partial_y + \frac{x^2}{2}\partial_z$$

of co-dim 1, is smooth and non-integrable because the Lie bracket

$$[X, Y] = x\partial_z \notin \Delta(x, y, z).$$

Remark 2
An integrable distribution is non-adjustable, i.e. it cannot be completed.

4. Horizontal path

Δ a smooth distribution of rank $m \leq n$ on M. A continuous path

$$\gamma(t) : [0, T] \longrightarrow \mathbb{R}^n$$

is said to be Horizontal with respect to Δ if $\gamma'(t)$ is absolutely continuous with square integrable derivative (i.e., $\gamma'(t) \in L^2([0, T], \mathbb{R}^n)$), and satisfies

$$\gamma'(t) \in \Delta\big(\gamma(t)\big),\ t \in [0, T] \text{ almost every where } T > 0. \text{ [1]}$$

We denoted by $\Omega_{\Delta}^{x,T}$, the set of Horizontal paths γ starting at x, for every $x \in M$ and $T > 0$.

Example 8
Let's consider the distribution

$$\Delta(x,y,z) = spam\left\{X = \partial_x; Y = \partial_y + \frac{x^2}{2}\partial_z\right\}$$

on $\mathbb{R}^3$. Let the continuous path

$$\gamma(t) : [0,T] \longrightarrow \mathbb{R}^3, \qquad t \rightsquigarrow \gamma(t) = (t,2,-1)$$

So it's clear that $\gamma'(t) = (1,0,0)$ is absolutely continuous in the sense of a vector function and $\gamma'(t)$ is square integrable on [0,1] because $\gamma'^2(t) = (1,0,0)(1,0,0) = 1 = Cste$, moreover

$$\gamma' = (1,0,0) = \partial_x \in \Delta(x,y,z).$$

Therefore, γ is a horizontal path with respect to the given distribution.

5. Totally nonholonomic distribution

Definition 3
Let M be a smooth manifold of dimension $n \geq 2$. A smooth distribution Δ on M of rank $m \leq n$ is said to be totally nonholonomic if

$$\forall x, y \in M, \exists \text{ a horizontal path connecting } x \text{ and } y. \text{ [1]}$$

Proposition - Characterisation
On smooth manifold M of dimension $n \geq 2$, a smooth distribution Δ on M is totally nonholonomic if and only if it is not integrable.

Proof - Example 9
In $\mathbb{R}^3$, the distribution defined by

$$\Delta(x,y,z) = spam\{X(x,y,z); Y(x,y,z)\}, \qquad \forall (x,y,z) \in \mathbb{R}^3$$

with

$$X = \partial_x \quad \text{and} \quad Y = \partial_y + \frac{x^2}{2}\partial_z$$

is totally nonholonomic because the Lie bracket $[X,Y] = x\partial_z \notin \Delta(x,y,z)$.

Example 10

Let's take $M = \mathbb{R}^3$, and two vector fields

$$X^1 = \partial_x - \frac{x^2}{2}\partial_y, \qquad Y = \partial_y + \frac{x^2}{2}\partial_z$$

then the distribution

$$\Delta(x) = spam\{X^1(x); X^2(x)\}, \qquad \forall x \in \mathbb{R}^3$$

is totally nonholonomic.

because,

$$[X, Y](x, y, z) = D_{(x,y,z)}X.Y_{(x,y,z)} - D_{(x,y,z)}X.Y_{(x,y,z)}$$

$$= \lim_{t\to 0} \frac{(e^{-tY} o\, e^{-tX} o e^{tY} o\, e^{tX})(x, y, z) - (x, y, z)}{t^2}$$

$$= -x\partial_z \notin \Delta(x, y, z) \qquad (2)$$

with e^{tX} and e^{tY} the waves of X and Y.

5.1 Degree of nonholonomy

Let Δ be a distribution of rank m on M. So for every point $x \in M$, we call the degree of nonholonomy of Δ of local frame $\Lambda = \{X^1, \cdots, X^r\}$ en x, the smallest integer $r = r(x) \geq 1$ such as

$$Lie^r \{X^1, X^2, \cdots, X^m\}(x) = T_x M. \quad [1]$$

with

$$Lie^r(\Lambda)(x) = \left\{X^1, \cdots X^r, \left[X^i, X^k\right], \left[\left[X^1, X^k\right], X^r\right], \cdots\right\}(x),$$

$$2 \leq i, k \leq r$$

Example 11

The distribution $\Delta(x, y, z)$ on $\mathbb{R}^3$ in examples 7 and 9 has degree 2 because for the first time

$$[X^1, X^2] = x\partial_z$$

on a

$$X^1(x, y, z), X^2(x, y, z), [X^1, X^2](x, y, z)$$

linearly independent, and :

$$Lie^r\{X^1(x,y,z), X^2(x,y,z), [X^1,X^2](x,y,z)\} = T_{(x,y,z)}\mathbb{R}^3,$$

$\forall (x,y,z) \neq 0_{\mathbb{R}^3}$

If, for example, we calculate again

$$[[X^1,X^2],X^2] = \partial_z \in \{X^1(x,y,z), X^2(x,y,z), [X^1,X^2](x,y,z)\},$$

then the distribution Δ has degree 3 on $\mathbb{R}^3$.

6. Structure sous-riemannienne

1. Let M be a smooth manifold of dimension n, Δ a totally nonholonomic distribution of rank $m \leq n$ and g a Riemannian metric (positive definite scalar product) on $\mathbb{R}^n$ containing M.

   The sub-Riemannian structure on M, is given by the distribution Δ_x and the metric g_x, i.e. the pair (Δ_x, g_x), for $x \in M$.

2. A sub-Riemannian manifold is the triplet (M^n, Δ^m, g) with $n \geq 2$ and $m < n$.

Example 12
All distributions $\Delta(x,y,z)$ on $\mathbb{R}^3$ given in examples 7, 9 and 10, which are totally nonholonomic, with the metric

$$g(X,Y) = \sum_i X_i.Y_i$$

for every $X,Y \in \Delta(x,y,z)$ confers in $\mathbb{R}^3$ the sub-Riemannian structure, hence a sub-Riemannian manifold.

Remark 3
The Riemannian manifolds $\mathbb{R}$ and $\mathbb{R}^2$ do not admit sub-Riemannian structures because all distributions on these spaces are integrable.

6.1 Sub-Riemannian structure on curves manifolds

Definition 4 :Non-degenerate 1-form
A smooth 1-form α is said to be non-degenerate if it does not cancel at any point $x \in M$
(i.e. $\alpha_x \neq 0, \forall x \in M$). Then $\forall x \in M$, the space $\Delta(x) = Ker(\alpha_x)$ is a distribution of co-dim 1 of M. [1, 5]

Remarks 4

1. Such distributions are often used to define (non-trivial) distributions in "curved" manifold.

2. It is also easy to define on these kinds of manifold M a non-trivial distribution with a 1-form if the latter is a contact form on M.

3. M is a contact manifold (i.e. M has odd dimension $n = 2p + 1$), a contact form α on M is a hyperplane field to each point $x \in M$ such that

$$\alpha \wedge (d\alpha)^p$$

is a volume form on M. [7, 8]

This means that, in a point $x \in M$, the kernel of α is a hyperplane of tangent space
T_xM,

$$Ker(\alpha_x) \subset T_xM$$

hence, a distribution on M.

Definitions 5

1. Let M a smooth manifold, Δ a distribution on M, E_0 a field of unit vectors of Δ and α a 1-form on M. We say that the metric g is associated to the 1-form α relative to E_0 if
$$\alpha E_0(X) = g(X, E_0), \quad \forall X \in \Delta. \text{ [3]}$$

2. Let M be a differential manifold, T_pM the vector space on $\mathbb{K}$ and T_p^*M its dual on

M. The duality product of T_pM and T_p^*M is the application
$$g: T_pM \times T_p^*M \to \mathbb{K}, \qquad (\alpha_p, X_p) \rightsquigarrow g(\alpha_p, X_p) = \alpha_x(X_p)$$
$\forall p \in M$. [2, 11]

Proposition 1
If the 1-form α is non-degenerate onThen the associated metric g is also M. degenerate. non-

Theorem 1
Let M a manifold of dimension $n = 2p + 1$ and α a 1-form on M satisfying

$$\alpha \wedge (d\alpha)p \neq 0$$

then the distribution given by

$$\Delta = Ker(\alpha)$$

is totally nonholonomic of degree 2. The 1-form α is the contact form and the associated distribution Δ is contact distribution. [1]

Proof
Giving yourself a point $\bar{x} \in M$, a local coordinate system exists $(x_1,\cdots,x_n)$ in the neighborhood ν of x such as

$$\alpha = \left(\sum_{k=1}^{2p} a_k dx_k\right) + dx_n$$

where $a_1,\cdots a_{2p}$ the smooth scalar functions on ν such as

$$a_k(\bar{x}) = 0, \qquad \forall k = 1, \dots 2p.$$

Thus, the family of smooth vector fields $\overline{X_1},\cdots,\overline{X_{2p}}$ given by :

$$\overline{X} = \partial x_k - a_k \partial_{x_n}, \qquad \forall k = 1,\cdots,2p$$

defines a local frame of $\Delta = Ker(\alpha)$ in ν. So the $n = 2p + 1$-forms $\alpha \wedge (d\alpha)^p$ in $\bar{x}$ is written :

$$(\alpha \wedge (d\alpha)^p)_{\bar{x}} = \sum_{\sigma \in P_{2p}} [\prod_{l=1,\dots,p} (\frac{\partial \alpha_{j_l}}{\partial x_{i_l}} - \frac{\partial \alpha_{i_l}}{\partial x_{i_l}})] \wedge dx_n \wedge \left(dx_{i_1} \wedge dx_{j_1}\right) \wedge \dots \wedge \left(dx_{i_p} \wedge dx_{j_p}\right)_{|\bar{x}}$$

such as $\sigma = \left((i_1, j_1),\cdots,\left(i_p, j_p\right)\right), i_l, j_l \in \{1,\cdots,2p\}$ and

$$[\overline{X^i}, \overline{X^j}](\bar{x}) = \left(\partial x_i\, a_j - \partial x_j\, a_i\right)\partial x_n\,(\bar{x}),\ \forall i, j = 1, \dots, 2p. \text{ [1]}$$

Theorem 2 (M. Makiese. K)
Any contact manifold M of dimension greater than or equal to 3 has a sub-Riemannian structure.

Proof - Indication :
On a contact manifold M, we have a contact form α which is generally a non-degenerate 1-form. Considering a Riemannian metric g in the space $\mathbb{R}^n$ containing M, knowing that

$Ker(\alpha) = \{X \in \chi(M) : \alpha_x(X) = 0, x \in M\}$ is a distribution on M and by doing the restruction of g on $Ker(\alpha)$, the pair $(Ker(\alpha), g_{|Ker(\alpha)})$ is a sub-Riemannian structure on
M.

Corollary 1 (M. Makiese. K)
Any hyperplane of the tangent space of a manifold M defined by the kernel of a nondegenerate 1-form on M with a Riemannian metric of the space $\mathbb{R}^n$ containing M, defines the sub-Riemannian structure on M.

Examples 13

1. Consider the unit sphere $\mathbb{S}^3$ of $\mathbb{R}^4$ as given :

$$\mathbb{S}^3 = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 | x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}.$$

et α a non-degenerate smooth 1-form on $\mathbb{S}^3$ (i.e. $\alpha_x \neq 0, \forall x \in \mathbb{S}^3$) defined by

$$\alpha = (x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)_{|\mathbb{S}^3}$$

then according the theorem 1 et 2, $\Delta = Ker(\alpha)$ is a totally nonholonomc distribution of degree 2 on $\mathbb{S}^3$.

Also, on Δ we have the smooth vector fields $X^i = \sum \partial x_i - a_{nk}\partial_{x_n}$ with a_{nk} smooth scalar functions.
Considering one of smooth vectors fields in the family $(X_i)_{i=1,2,3,4}$ given by :

$$X = x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2}$$

in Δ for which :

$$\begin{aligned} g_\alpha &= g(\alpha, X) \\ &= \alpha(X) \\ &= (x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)(x_1 \tfrac{\partial}{\partial y_1} - y_1 \tfrac{\partial}{\partial x_1} + x_2 \tfrac{\partial}{\partial y_2} - y_2 \tfrac{\partial}{\partial x_2}) \\ &= x_1^2 + y_1^2 + x_2^2 + y_2^2 \end{aligned} \quad (3)$$

So we see that g_α is a Riemannian metric relative to the chosen vector field and of the form α on $\Delta = Ker(\alpha)$.
Thus, $(\mathbb{S}^3, Ker(\alpha), g_\alpha)$ is a sub-Riemannian manifold.

2. Let the Torus $\mathbb{T}^3$ of $\mathbb{R}^4$. Knowing the contact form on T^3 given by the equation :

$$\alpha(x_1, x_2, \theta) = cos\theta dx_1 + cos\theta dx_2 = 0,$$

$(x_1, x_2) \in \mathbb{R}^2/\mathbb{Z}^2, \theta \in \mathbb{R}/2\pi\mathbb{Z}$. [8, 7]
Now, $\forall x \in \mathbb{T}^3$, the form α is a hyperplanes field on $\mathbb{T}^3$, so at a point $p \in \mathbb{T}^3$, $Ker\alpha_p$ is a hyperplane of $T_p\mathbb{T}^3$, i.e.

$$Ker\alpha_p \subset T_p\mathbb{T}^3$$

and according to Theorem 1 and the previous remarks, $Ker\alpha_p$ is a totally nonholonomic distribution on $\mathbb{T}^3$.
So, if g is a Riemannian metric of $\mathbb{R}^4$, then any restriction g_x on $Ker\alpha_x$ to each point $x \in \mathbb{T}^3$ gives the pair $(Ker\alpha_x, g_x)$ the sub-Riemannian structure.

Thus, $(\mathbb{T}^3, Ker\alpha, g_\alpha)$ is a Sub-Riemannian manifold.

4. Conclusion

Sub-Riemannian geometry generalises Riemannian geometry, especially since to a point x of a manifold M^n, a totally nonholonomic distribution to that point is a tangent subspace whose dimension belongs to $[1, n[$.

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