ENHANCING STUDENTS' MATHEMATICAL REPRESENTATION ABILITY THROUGH MATHEMATICS LEARNING

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Abstract

Several previous researchers indicated that the students' mathematical representation ability is poor due to the fact almost students think that mathematics may be very hard, dull, hard to sensible, and has too many abstract theorems which are very difficult in recognizing. Another reason is the teacher still uses the mechanisms technique in the learning process without deliberating students' requirements. Learning is over notable as transferring the knowledge process to students. Allow students to build up their understanding by the concrete and mental reflection that is retrieved with an action in the new expertise. This article is literature research. This article aims is examining the Concrete-Pictorial-Abstract approach theoretically to enhance students' mathematical representation ability. The result of this research is advancing the students' mathematical representation skills can use the Concrete-Pictorial-Abstract approach.

Keywords: *The Concrete-Pictorial-Abstract approach; mathematical representation ability; literature study.*

1. INTRODUCTION

Representation is mathematics education that uses the concept of psychology to describe any essential indication approximately the process of the thinking students which is represented in language, visible and symbolic forms. Goldin said Representation is one of the configurations or forms, characters, symbols, or objects that may describe, represent or symbolize the other forms [1]. Increasing the students' representation is the expressiveness of the mathematical thought offered by them as a sample to get away from the matter they are dealing with [2]. Attainment of mathematics capability needs verbal exchange or forms of writing that in the simple or more than

one representation configuration arranged in the language of mathematical [3]. Representation capability is required in the development and attends to the students' understanding of the draft [4]. The representation used to expand students' information is closely associated with the capacity to do the operation with their representations [5].

The problem-solving abilities of students are also closely connected to their representation ability. Hwang [6] stated that one of the keys to success in solving the problem is having multiple representations. The best problem solver generally constructs the problem's representation as a settlement in making easy perceptivity [7]. Sajadi [8] appointed that problem-solving was not successful without initial right representations of the problem. Moreover, he said that students will have complicated in solving their mathematics problems if they can't make an illustration first. The biggest inconvenience in process of problem-solving happened at the illustration stage [9]. The best strategy to fix the mathematics problem is to integrate the illustration capability and the manipulate symbol capability [10][11]. When students can recognize the diverse forms of mathematical illustration, they should be able to realize the mathematical concept utilized in solving the problems.

Refers to the expert's explication, it could be deduced that the mathematical representation skill is significantly for student inasmuch as an elementary for other mathematical abilities as well as the understanding ability concept, communication abilities, problem-solving skills, reasoning skills, and so on.

Based on PISA 2018 [12], Indonesian students attained lower scores than the OECD's average in mathematics. Only about 28% of Indonesian students can reach level 2 whereas the OECD average is 76%. Students may interpret and recognize, devoid of immediate instruction how modest fettle could be represented mathematically is

the indicator at OECD test. It verifies that the mathematical representation skill of the Indonesian students is relatively poor [13][14][15].

The low mathematics representation skill of the students in junior high school since they are no longer allowed in building and presenting their representations connected to the subject being studied [16]. In the meantime, students from the 11th to 14th years old, construct the development theory by Piaget are at the initial formal operation phase. The characteristic of this phase is the transition of students' experience through the concrete thinking phase to the abstract thinking phase. Representation can use to bridge this phase because students are not yet completely capable of thinking abstractly. This illustration skill will bring students to more abstractly thinking ability. Even, follow to Dahlan [17], the poor students' mathematics illustration ability is in consequence of the design of the learning mathematics process from the teacher which is almost deductive in its practice, where students are given the mathematical formulas, rules, or propositions without preparing contextual problem connected to the subject being taught.

Hence, students require a learning approach that may offer the possibility to discover their skills, establish, and bring out their representation connected to the subjects they have learned. In addition, students require learning that starts by supplying the contextual problems connected to the topic before going up to the abstract phase. Consequently, students can acquire the mathematical representation potential standards.

Choose an approach shall be adapted for students' thinking development phase, so the learning process is more significant, and fun, and make students overwhelmed vigorously in retrieving their perspectivity and awareness. The Cognitive development theory by Piaget [18] suggests four thinking stages for every person in accepting knowledge, that are (1) the sensorimotor term (2) the pre-operation term (3) the concrete operational term (4) the formal operational term.

The way that can be used to advance students' mathematical illustration ability is by selecting a proper learning approach. Either of the approaches may advance students' mathematical representation skills to actively complicity in retrieving their concepts and knowledge learned by the Concrete-Pictorial-Abstract approach. The goal of this article was to discuss the relationship between the Concrete-Pictorial-Abstract approach and the students' mathematical representation ability, particularly the in the middle school students.

2. RESEARCH METHOD

This literature research was examining in theoretically about the Concrete-Pictorial-Abstract approach in advancing students' mathematical representation ability. This research exploits five theories of mathematical representation: [2][18][19][20][21] and six research articles using the Concrete-Pictorial-Abstract approach [5][6][22][23][24][25]

3. RESULT AND DISCUSSION

1. Result

1.1 Mathematics Representation Ability

Representation is a way to communicate the idea, notion, or mathematics answer [2] Representation is a part of communication that may be in the first language form, mathematical verbal language, visual representation, and quasi-mathematical language [3] Mathematics representation is an ability to represent notation, symbols, diagrams, graphs, and mathematics equations into other forms.

Mathematical representation has a very important job in various activities of numerical and is a crucial part of the literacy of mathematics which is the goal of education in various countries [10][11],as equipment for manipulating objects, a communication tool, and as equipment for appreciating concepts [26]. Furthermore, representation supports mathematical reasoning, mathematical communication, and conveying mathematical ideas or ideas [21].

The teacher can use representation to teach. Likewise, students can use it in solving problems and communicating their ideas of mathematical to others. When students are confronted with a mathematical problem, the first thing to do is try to comprehend the matter and solve it in a way that has been understood. One of the best ways is to create a model or representation of the problems. The representation is made based on the students' capabilities in managing problems.

Representation competence concerns: the ability to understand mathematical ideas in various forms, understand when and why a situation can use certain mathematical representations, can translate ideas into representations, and use representations flexibly to solve problems [20]. *Standards and Positions Process* in NCTM about students' representation ability, which can:

1. Build and apply representation to conduct, remind, and recite the ideas of mathematics.
2. Choose, occupy, and analyze various mathematical representations in solving the problems
3. Apply the representation to design and represent physical, social, and mathematical phenomena.

Bruner distinguishes three types of representation that are closely related to the child development stage, namely: enactive mode, specifical representation through action; iconic mode, specifical representation through visual images or schematics; and symbolic mode, specifical representation through words and language [19]. Salkind [4] discovered five varieties of representation which are the development of Burner's representation theory used in mathematical understanding: (1) The real experience (2) Manipulation of models (3) Pictures or diagrams (4) Words or language (5) The symbols. The real experiences and manipulation models are part of the enactive representation, pictures and diagrams are iconic representations, and written language and epitome are symbolic representations.

Several methods of mathematics representation are used in problem-solving: (1) numerical representation such as decimal, fraction, or percent (2) graphical representation which includes six different visual representations, that are: pictures, models, horizontal schematic, vertical schematic, charts, and coordinate graphs (3) verbal representation which involves using the written language to conceive, explain, elaborate and clarify the steps in problem-solving (4) symbol representation which focuses on symbol notation, using of variables and formulas [10].

Huinker [20] distinguishes representation into five types: *"visual representation, verbal representation, contextual representation, physical representation, and symbolic representation".*

1. Visual representation is an ability to illustrate, demonstrate or work with ideas of mathematics using diagrams, graphs, number lines, and other math illustrations.
2. Verbal representation, use language in interpreting, discussing, defining describing mathematical ideas; connecting informal and formal mathematical language.
3. Contextual representation, situate mathematical ideas in daily life, describing situations, using a variety of discrete and continuous (such as people, meters, yards).
4. Physical representations, use real objects to demonstrate or manipulate mathematical ideas (such as cubes, counters, tiles paper strips).

5. Symbolic representation, recite or work with the ideas of mathematics using numerals, variables, and other symbols.

Wessel [27] divides representation into five types: language representation, arithmetic symbol representation, picture or graphic representation, real-world object representation, and concrete representation. The first three of them are the highly abstract and high-level representations using for mathematical problem-solving.

Goldin dan Shteingold [28] divide two representation system. External representation patterns contain traditional representations usually symbolic in nature. Notation systems, natural language, visual imagery, and problem-solving tricks are part of external representation. Internal representation systems are made inside a person's mind and used to establish mathematical meaning. Algebraic expressions, mathematical equations, numeration systems, graphs, geometric figures, and number lines are examples of external representations. Development of this representation has been done a long time ago and used considerably

Build upon the classifications above, it can be terminated that representation ability includes: 1) Visual representation such as picture, diagram, graph or table form 2) Symbolic representation such as mathematical notation, numerical or algebra symbols 3) Verbal representation such as text or spoken language.

Some benefit of learning involves representation for teacher and student are [29]:

1. Provide many contexts for learning
2. Enhancing students' understand
3. creating representation a conceptual utilize
4. Improve student's ability to use mathematical representations as a problem-solving tool
5. Minimize the misconception occurrence

1.2 The *Concrete-Pictorial-Abstract* (CPA) Approach

The Concrete-Pictorial-Abstract (CPA) or the Concrete-Representation-Abstract is an approach repose Bruner's heuristic cognition about the enactive, iconic, and symbolic method of representation been distinguished by instructional supported from the Education of Singapore's Ministry ever since

the early 1980s [23]. The CPA approach supports the basic understanding of mathematical concepts before learning more complex mathematical rules that move from the real or manipulative form of blocks or multiplication going up to the abstract representation as well as 2 x 4 = 8 [29]. The Concrete-Pictorial-Abstract approach will grant an advantage to students who have problems in the mathematics learning process because its approach started using real materials, followed by learning through pictorials, and ends by using abstract symbols [22].

By this approach, students start to learn by using a touchable object, representations of pictorial, and abstract numerals at the end. The Concrete-Pictorial-Abstract approach is also known as a three-phase learning process where students in learning using the touchable object, attended by learning through pictorial representations of the concrete objects, and ended with using the abstract notation in solving [24].

The third phase is a cohesive whole that involved and connected one another in practice. Elements of the concrete contain manipulation, using the equipment, or other objects touchable during the learning process. Manipulatives are vulnerable because they are concrete and touchable. This sensory makes manipulatives feel so 'real,' concerned with the individual to make meaningful intuitively. intuitively meaningful personal self, and therefore helpful [30]. The pictorial elements are used for students in reading and interpreting. It's cover drawings, paintings, diagrams, or graphics created or provided for students to read and interpret. Whereas the abstract phase uses symbols, numbers, or letters in solving the mathematics tasks.

The level of learning in every phase of this approach is described below:

a) Concrete level
- Students are offered or build their manipulative objects connected with the concept to be learned
- Teachers provide verbal explication and questions by demonstrations
- Students start to think with manipulative object

b) Pictorial level
- Students create representations concerned with geometric drawings, graphs, or charts that may reflect a formerly used manipulative object

- Students are provided a series of questions related to the representation's form of the manipulative object.

c) Abstract level

- Students discover an arrangement from a concept have been learned using symbols or abstract mathematical language
- Students are provided some queries to train their mathematics skills using abstract symbols in solving problems.

Detail of learning by the CPA approach's steps are [25]:

1) Select the manipulative objects that would be used to present a conceptual understanding
2) Students are guided to contribute in using the manipulative, providing prompting and signing, and students independently occupy manipulative objects to testify their skills
3) Switch the manipulative object with pictures and/or drawings
4) Teachers utilize certain tactics that may assist students in reminding the steps embroiled in the mathematical ability. This phase is a switchover from pictures or drawings using to numbers only using. We called this phase as the abstract phase
5) Students apply for just numbers when finishing the task, and instruction focuses on continuity.

Several research supports the efficacy of this approach. Students in levels VII and VIII indicated inconvenience in algebra and acquired the top outcome when learning using the Concrete-Pictorial-Abstract approach in solving the algebra equations transformation more than the students who learn by conventional instruction [24]. furthermore, students perform few errors when solving the algebra problem using the sequence procedures of the Concrete-Pictorial-Abstract approach [22]. Learning mathematics by the Concrete-Pictorial-Abstract approach gives some benefits: (a) may expand a real understanding of the mathematics concepts or abilities they have learned (b) may apply this basic and increase their understanding of conceptual to abstract problems (c) Understanding of the mathematical concept and idea are had in deeply and offer an extraordinary strategy for problem-solving in other fields in the future [31].

The Concrete-Pictorial-Abstract approach is a structure that simplifies students to create purposeful relationships among the concrete object, and the understanding degree uses pictorial and abstract thinking. It's because learn

initiated through visual experience, substantive and kinesthetic to construct a primary understanding. Afterward, students may develop their knowledge via pictorial representation (pictures, sketches, or diagrams) and eventually switch to the level of abstract thinking. Students exclusively may employ mathematics symbols for representing and modeling the issues connected to the material had been learned.

2. Discussion

Mathematical representation skill is an ability that allows students to relate mathematics learning to the fourth main aspect: visual, image, mathematical equations or expressions, and written text. The representation skills are the foundation for other mathematical abilities like understanding concepts, communication skills, reasoning skills, and problem-solving skills. Students with high mathematical representation ability will have no difficulty in solving mathematical problems. High students' mathematical representation ability is affected by the teacher and the approach used in the process of learning.

In a way to enhance students' mathematical representation skills is needed the learning approach that might supply changes for students to investigate their skills, build, and provide their own representation connected to the subject they had learned. In addition, it starts with giving context related to the subject before moving to the abstract level. Students may attain the mathematical representation skill standards.

Choose an approach must be adopted by the stages of students' thinking development so learning activities are over significant, fun, and makes students vigorously embroiled in retrieving their perspectivity and awareness. There are four phases in hierarchical learning: (1) concrete stage (2) semi-concrete stage (3) semi-abstract stage (4) abstract stage. This is according to the cognitive development theory by Piaget which recommends four levels of thinking for individuals in accepting knowledge: (1) sensorimotor stage (2) pre-operation (3) concrete operational stage (4) formal operational stage [18]. Another alternative learning which refers to the students' thinking levels and involves students vigorously constructing their perspectivity and awareness is using the Concrete-Pictorial-Abstract approach in the learning process.

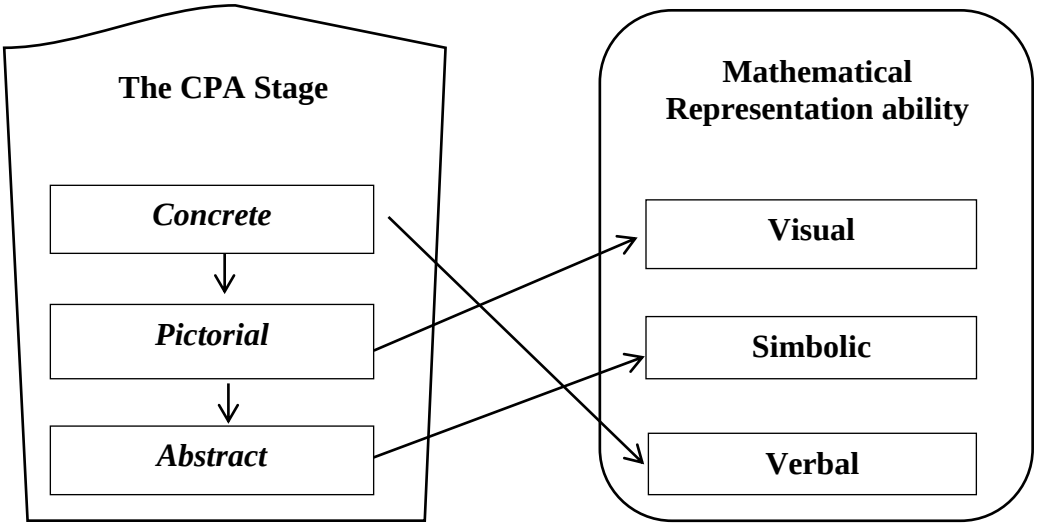
The *Concrete-Pictorial-Abstract* approach is one of the forms of constructivism learning, where learning started by using the manipulative objects connected to the subjects to search and discover a concept. This stage gives students a chance to realize that mathematics is closely to their daily life and feel the profit from learning mathematics in solving the problem.

The second stage is pictorial. This phase is like a bridge that can associate students' initial knowledge who just requires concrete objects, toward the abstract learning phase including numbers, symbols, and notations. Students have a chance to draw concrete objects that resemble their original form, then proceed with symbolic images. For example, when students learn to find the properties of a rectangle, they can observe objects around which like a rectangle form, then draw them. Through the pictures they have made, students can find and write the properties of rectangles in an abstract way. The last stage is the abstract. Focus on this phase is students' fluency in solving mathematical problems using symbols, notations, or numbers only.

Students' thinking development level is noticed in this approach, particularly in second-class students whose ages are from 11-14 years old. Referring to Cognitive Theory by Piaget, ages from 11 to 14 years old begin the formal operational phase, where students start thinking abstractly, logically, and ideally even though not maximally because they are in the switchover phase between the concrete thinking phase to the abstract thinking phase. accordingly, the Concrete-Pictorial-Abstract approach is equipment in bridging that thinking phase.

Learning mathematics through the CPA approach provides opportunities for increasing students' mathematical representation abilities because the CPA approach refers to the students' thinking stages development and reflects the active entanglement of students in retrieving understanding and knowledge. Besides that, this approach facilitates students to make meaningful connections among concrete, pictorial, and more abstract levels of understanding and thinking. In this phase, students begin learning using visual, real, and kinesthetic experiences to construct a prior understanding afterward, students may develop their awareness through pictorial representations (pictures, diagrams, or sketches) and finally may devolve to the level of abstract thinking, where students can apply the mathematical symbols in representing and modeling problems connected to the material they had learned. exclusively

The CPA approach that is carried out in stages can enhance students' representation abilities since learning begins from the easy stage, namely the Concrete stage, and then slowly switches to more complex stages. This is similarly expressed by Visscher [33] that one of the foundations of learning theory is learning that starts from the simplest stages to more complex stages. The learning process by the CPA provide many occasions for students to have and enhance their mathematical representation abilities.



Pigure 1 Scheme Framework

4. Conclusion

Mathematics representation ability as a way to understand the concept and communicate mathematics ideas shall be enhanced by using the Concrete-Pictorial-Abstract approach. This approach may help students to construct understanding and knowledge about a mathematics concept. Learning mathematics by the Concrete Pictorial Abstract approach is in accordance with grade VII of students' thinking stage who still need object concrete before going to an abstract level.

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